Self-similarity of two flows induced by instabilities

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The implications of full self-similarity of the Rayleigh-Taylor mixing layer and the Kelvin-Helmholtz shear layer are examined using a simplified group-theoretic analysis. The constraints on the behavior and evolution of these layers imposed by rigorous self-similarity are identified, and equations are constructed for the growth rate of these layers based on a total energy balance. This analysis does not prove that such flows will become self-similar. Rather, the analysis demonstrates the behaviors that would arise if these flows were to become fully self-similar.

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I. INTRODUCTION AND BACKGROUND ANALYSIS

Flows induced by instabilities are found in many engineering and astrophysical circumstances. Specifically, the instabilities induced by acceleration (or gravity) and shears have attracted much attention and carry the well known names of Rayleigh-Taylor (RT) [1,2] and Kelvin-Helmholtz (KH) instabilities [3]. While the initial linear, nonlinear, and transient processes are complicated, it is widely suspected that at late time the RT and KH flows will relax toward a self-similar statistical state where the dominant length scale, i.e., the mixing-layer width is growing as an algebraic function in time. The late-time scaling is typically based on physical arguments and experimental and numerical results appear to support the self-similarity assumption, at least in so far as the evolution of the mixing-layer width is concerned. More precisely, the fundamental assumption of self-similarity is that a RT or KH configuration starting from an arbitrary initial state will relax toward a solution of the evolution equations that is invariant under an appropriate symmetry group.

The purpose of this paper is not to prove or disprove the actual existence of such a self-similar state for a particular physical system or for a mathematical model of such a system. Likewise, our purpose is not to demonstrate the attainment of a self-similar state in experiments or computations. Rather, our goal is to elucidate the nature of a fully self-similar state for such systems. With this limited goal in mind, it is still useful to note that the expectation is that the physical system (e.g., a RT or KH mixing layer) would tend toward the self-similar state at late times, after an initial transient time in which the correlations of fluctuating quantities become established. The attainment of late-time solutions, either numerically or experimentally, is a daunting task for processes which are growing as power laws in time. Determining whether a system failed to achieve self-similarity due to simulation times that are too short, simulation sizes that are too small, or for some fundamental physical cause requires understanding the nature of the anticipated self-similar state. Our goal is to describe a self-similar state consistent with the mathematical requirements of self-similarity and consistent with the physical processes active in these flows.

In this paper, we turn our attention to investigating the late-time, highly chaotic state that occurs when the flow field is induced by Rayleigh-Taylor and Kelvin-Helmholtz instabilities. Both flows are characterized as inhomogeneous in one dimension and homogeneous in the remaining two, and both possess a persistent energy source term—i.e., the mean shear in the KH layer, and the potential energy in the RT layer. Our primary objective is to determine the time-scaling laws of various statistical parameters that may arise in the study and modeling of these systems. A secondary goal is to determine the functional structure and dependencies of the growth rates of these flows. The methodology is a generalization of the approach recently applied to isotropic turbulence [4]. In addition, we will exploit the detailed energy balance equation among kinetic energy, potential energy, and their dissipation rates to construct expressions for the growth rate of these mixing layers, detailing the functional form and dependencies of the coefficients. The consequences of ''turbulencelike'' assumptions for the dissipation rate will be demonstrated. Furthermore, the late-time scaling properties of other important measurements are obtained.

The term ''turbulence'' in the context of the RT and KH mixing layers warrants explanation. For the circumstance of a multimodal initial interface perturbation, the RT mixing layer grows, after the initial ''linear'' growth regime into a chaotic flow of interacting ''fingers'' or ''mushrooms'' of penetrating fluids [5,6]. At late times, as the kinetic energy of the flow grows the flow in the core of the mixing layer looks increasingly chaotic. However, at the edges of the mixing layer the layer may still be characterized by relatively isolated fingers of fluid penetrating a relatively quiescent fluid field and thus may not appear to be classically turbulent. This distinction of a chaotic core flow and a strongly ''intermittent'' edge flow also pertains to the KH mixing layer. Recently, some researchers have attempted to define a ''mixing transition'' at which the fine scales typically associated with high-Reynolds-number turbulent flows emerge [7]. The emergence of these fine scales may mark the broadband spectrum of turbulence dominated by inertial range dynamics. In the context of the RT and KH mixing layers, the distinction between the dynamics of the core flow versus the dynamics of the edges suggests that this emergence of fine scales may occur at different times for different locations during the evolution of the flow. The present analysis does not explicitly pertain to the existence of an inertial range, or the lack of one. However, an assumption regarding the nature of the dissipative processes is made, and will be discussed in detail. The specific changes in scaling behavior that may or may not occur when inertially driven dissipative processes dominate over simple viscous processes will be addressed.

The self-similar growth of chaotic RT mixing layers and KH shear layers has been studied over the past 30 years, beginning with the seminal work of Taylor [2]. Various theoretical model descriptions, computational simulations and experiments have been conducted. The physical picture that has arisen is described by Sharp [5] and Youngs [6]. In the present circumstance, an initially randomly perturbed interface grows, first through an early stage which yields to linearized analyses, and later through a late-time, highly chaotic, nonlinear state. In the case of the RT mixing layer, the driving energy is the potential energy of the unstably stratified density field. In the case of the KH mixing layer, the driving force is the difference in velocities across the mixing layer.

Our approach to studying this evolutionary picture is to exploit the implications of self-similarity of these highly chaotic fluid layers. Decaying isotropic turbulence was studied by Clark and Zemach [4] using spectral closure models and a simple group-theoretic approach. Although the results achieved were not new, the approach used demonstrated that these previously reported results were consistent with the underlying dimensionality and symmetries of the functions studied, rather than a unique product of specific physical models or assumptions of the previous analyses. We will apply this same group-theoretic approach to these two simple mixing layers. First we will demonstrate the application of these ideas to the evolution of the mixing-layer widths. Next we will apply them to some fundamental ''turbulence'' quantities in these flows. We will then use these results to construct energy balances from which we will determine the governing equations governing the self-similar growth rate coefficients for these layers if a self-similar state is achieved. Finally, we will apply the methodology to investigate the self-similarity of an inhomogeneous spectrum. The results will not prove the existence of a self-similar turbulent growth. Instead, they will describe the behavior of a self-similar flow if such a flow were to exist in the context of RT and KH mixing layers. As such, these results will merely provide a basis for objective comparisons of real flows to a postulated mathematically self-similar turbulent flow.

II. SELF-SIMILARITY OF THE MIXING LAYER

It is widely believed that many simple turbulent flows, such as the RT and KH mixing layers evolve toward self-similarity. Youngs [6] exploited a self-similar modal growth of the RT interface to arrive at the now famous formula for mixing layer growth formula;

$$W(t)=\alpha A g t^2. \tag{2.1}$$

It must be noted that even earlier Sharp and Wheeler [8] (see also Sharp [5]) constructed a bubble amalgamation model for the RT that predicted an average velocity of bubble rise, v_{av}, of

$$v_{\mathrm{av}}(t)=k_1 g t, \tag{2.2}$$

where

$$k_1 \propto \frac{1}{20} \sim \frac{1}{100}. \tag{2.3}$$

The coefficient k_1 is related to Youngs α parameter for the bubble side of the layer and the Atwood number, e.g., $k_1 = 2\alpha_{\mathrm{bubble}} A$.

Attempts have been made to assess the degree of self-similarity of numerical simulations, e.g., [6,9,10], and a variety of models of RT mixing layers, e.g., [11–14].

Another approach is to substitute dimensionless functions in conjunction with dimensional time-dependent scaling factors into the governing equations, and then determining the necessary consistency relations for self-similarity. Ristorcelli and Clark [15] analyzed the second-moment equations of the Navier-Stokes equations in the Boussinesq limit for the Rayleigh-Taylor mixing layer, and compared the resultant scalings to direct numerical simulations. An interesting extension of this approach was exploited by Chen *et al.* [16] for compressible Rayleigh-Taylor mixing layers and by Cheng, Glimm, and Sharp [17] in the context of a bubble merger model. In these works, the scaling group was presumed, the equations were subjected to the scalings, and fixed-point analyses were carried out on the scaled system. This has proven to be a powerful tool, and provides further motivation for the present effort to clarify the mathematical basis of the appropriate scaling groups. Of course the KH mixing layer will yield to a similar analysis.

The intention of the present paper is to provide a more general self-similar description of self-similar mixing layers that is not dependent upon a particular mathematical model of the flow. With this goal in mind, it is expected that the results of the present analysis should agree with previous analysis if the underlying mathematical models used in previous analyses do not violate the underlying mathematical symmetries. The more general approach is to exploit the underlying scaling groups that define the self-similarity. This latter method owes much to the work of Lie, although the present use of this approach is greatly limited for utilitarian purposes. It requires no detailed physical model, or governing equation, relying instead on the consequences of self-similarity of the functional forms. The underlying viewpoint is that the physical system acts dynamically upon an initial state, and under this dynamical action the system relaxes to an invariant state. By invariant state, we mean a state that is invariant under appropriate classes of subgroups of the full group of space-time symmetries. This approach is described more fully by Clark and Zemach [4] and the interested reader is directed to that paper for a more complete description of the approach. To demonstrate the approach, we will consider the case of the mixing layer width $W(t)$. After this brief demonstration, we will consider the cases of inhomogeneous energy spectra, density fluctuation spectra, and other statistical characterizations of the flow.

A. Self-similar growth of the mixing layer—a detailed calculation

In order to determine the self-similar form of the Rayleigh-Taylor mixing layer or the Kelvin-Helmholtz shear layer, we begin by reviewing the scaling group exploited by Clark and Zemach [4] for the case of isotropic turbulence. Indeed, this group yields a power-law behavior in time, as observed for RT mixing layers and KH shear layers. These scalings are as follows: for a length scale, l scales to $\hat{l}$ by

$$\hat{l} = \lambda l, \tag{2.4}$$

and for a time scale, t scales to $\hat{t}$ by

$$\hat{t} = \tau(t+t_0) - t_0. \tag{2.5}$$

Note that we have included a translation in time t_0 and a rescaling of time τ and length λ. Self-similarity assumes

$$\lambda W(t) = W(\tau(t+t_0) - t_0). \tag{2.6}$$

We anticipate that the solutions will be in terms of power laws (this may be deduced from a classical similarity analysis, and is also the generally accepted notion of the behavior of the RT and KH mixing layers). For this case we will restrict the group as follows:

$$\tau^{\gamma} = \lambda, \tag{2.7}$$

so

$$\tau^{\gamma} W(t) = W(\tau(t+t_0) - t_0). \tag{2.8}$$

Differentiating with respect to τ yields

$$\gamma \tau^{(\gamma-1)} W(t) = \frac{dW(t)}{dt}(t+t_0). \tag{2.9}$$

Now set $\tau = 1$ to give the determining equation

$$\gamma W(t) = \frac{dW(t)}{dt}(t+t_0). \tag{2.10}$$

The solution of this ordinary differential equation is

$$W(t) = W_0 \left[\frac{t+t_0}{t_0}\right]^{\gamma}, \tag{2.11}$$

where t_0 may be positive or negative. It is assumed that this form applies for the case $t \gg t_0$. We now may restrict the subgroup dependent on the relevant physical parameter in the given flow field. For the case of a Rayleigh-Taylor mixing layer, the dominating physical parameter is assumed to be acceleration g, having dimensions of $[LT^{-2}]$. Under the scaling group above, we find that the acceleration scales as

$$\hat{g} = \tau^{\gamma-2} g, \tag{2.12}$$

which is invariant if $\gamma = 2$. We conclude that a self-similar theory for $W(t)$ that has a physical parameter with the dimensionality of acceleration will have the form

$$W(t) = W_0 \left[\frac{t+t_0}{t_0}\right]^{2}. \tag{2.13}$$

If the dominant physical parameter has dimensions of a velocity $[LT^{-1}]$, e.g., the KH shear layer where the free-stream velocities are U_1 and U_2, and the velocity difference across the layer

$$\Delta_u = U_2 - U_1 \tag{2.14}$$

is the dominant parameter, then we require that something with dimensions of velocity scale as

$$\hat{\Delta}_u = \tau^{\gamma-1} \Delta_u, \tag{2.15}$$

yielding $\gamma = 1$, or a relationship linear in time:

$$W(t) = W_0 \left[\frac{t+t_0}{t_0}\right]. \tag{2.16}$$

If viscosity is the dominant physical constant, then we find the group element from

$$\hat{\nu} = \tau^{2\gamma-1} \nu, \tag{2.17}$$

giving $\gamma = 1/2$ and

$$W(t) = W_0 \left[\frac{t+t_0}{t_0}\right]^{1/2}. \tag{2.18}$$

The same result applies if diffusivity is the dominant physical constant. If a dominant fixed length scale $\mathcal{L}$ is dynamically important, e.g., the size of a test vessel, or a fixed length scale in a theory or model, then

$$\hat{\mathcal{L}} = \tau^{\gamma} \mathcal{L}, \tag{2.19}$$

giving $\gamma = 0$ and

$$W(t) = W_0. \tag{2.20}$$

These results are not based upon any assertion regarding the detailed physical characteristics of the flows (e.g., turbulent, laminar, or other). Rather, they represent a mathematical statement regarding the form of self-similarity under a postulated physical constraint such as acceleration. Indeed, as can be seen from these results, the subgroup element that reconciles acceleration ($\gamma = 2$) is inconsistent with the subgroup element that reconciles viscosity ($\gamma = 1/2$) or that which reconciles a velocity difference ($\gamma = 1$). A real flow possesses both viscosity and an acceleration for the case of the RT mixing layer, or a velocity difference for the KH mixing layer. This implies that a concise single-parameter scaling group cannot describe all scales of the flow. This is perhaps relevant to the observations of Dimotakis [7] that the fine-scale ‘‘turbulent’’ mixing scales emerge at a specific Reynolds number, and are not a gradual manifestation of increasing Reynolds number. Likewise, Clark and Zemach demonstrated that the scaling group elements that reconciled the largest scales of an isotropic turbulence were not necessarily consistent with those applicable to the dissipative scales. For the present paper, we will restrict our attention to the accelerative case (RT with $\gamma=2$) and the velocity-difference case (KH with $\gamma=1$). These group elements effectively exclude self-similarity of the viscous scales in a flow field. That is to say, the presence of viscosity in the KH shear layer or RT mixing layer may preclude absolute self-similarity in the same way that it excludes full self-preservation in the case of isotropic turbulence.

B. Self-similarity of density across the layer

The mean (planar-averaged) density $\overline{\rho}(z,t)$ across the RT or KH mixing layers is bounded by ρ_2 (the heavy fluid) and ρ_1 (the light fluid) for the case of incompressible flow. Note that the density has units of mass per unit volume, e.g., $[M/L^3]$. Assuming that

$$\hat{M}=\tau^{\beta}M, \tag{2.21}$$

we have

$$\tau^{\beta-3\gamma}\overline{\rho}(z,t)=\overline{\rho}(\tau^{\gamma}z,\tau(t+t_0)-t_0). \tag{2.22}$$

An alternative approach would be to assign the density a scaling parameter, say $\beta_0=\beta-3\gamma$. This present approach to determining the self-similar forms would yield a similar conclusion for these incompressible flows. Since $\overline{\rho}$ is a function of two variables [unlike $W(t)$], the details of the solution will be presented for clarity. First differentiate with respect to τ and set $\tau=1$, and divide through by $t+t_0$:

$$\frac{\beta-3\gamma}{t+t_0}\overline{\rho}=\frac{\gamma z}{t+t_0}\frac{\partial\overline{\rho}}{\partial z}+\frac{\partial\overline{\rho}}{\partial t}. \tag{2.23}$$

The characteristic equation for z is

$$\frac{dz(t)}{dt}=\frac{\gamma z}{t+t_0}, \tag{2.24}$$

with the solution

$$z(t)=z_0\left[\frac{t+t_0}{t_0}\right]^{\gamma}. \tag{2.25}$$

Substituting gives

$$\frac{\beta-3\gamma}{t+t_0}\overline{\rho}=\frac{dz}{dt}\frac{\partial\overline{\rho}}{\partial z}+\frac{\partial\overline{\rho}}{\partial t}=\frac{D\overline{\rho}}{Dt}. \tag{2.26}$$

The solution is

$$\overline{\rho}(z(t),t)=\overline{\rho}_0(z_0)\left[\frac{t+t_0}{t_0}\right]^{\beta-3\gamma}$$
$$=\overline{\rho}_0\left(z(t)\left[\frac{t+t_0}{t_0}\right]^{-\gamma}\right)\left[\frac{t+t_0}{t_0}\right]^{\beta-3\gamma}, \tag{2.27}$$

or equivalently

$$\overline{\rho}(z,t)=f_{\rho}\left(\frac{z}{W(t)}\right)\left[\frac{t+t_0}{t_0}\right]^{\beta-3\gamma}=f_{\rho}(\chi)\left[\frac{t+t_0}{t_0}\right]^{\beta-3\gamma}, \tag{2.28}$$

where

$$\chi=\frac{z}{W(t)}. \tag{2.29}$$

Clearly, the function varies from ρ_1 on one side of the mixing layer to ρ_2 on the other. These densities are constant in time and imply that $\beta_0=\beta-3\gamma=0$. This implies that the mass in a volume scales with the volume and is a corollary to the incompressibility assumption. So the form for the density is

$$\overline{\rho}(z,t)=f_{\rho}(\chi). \tag{2.30}$$

The density field may also be represented as a concentration function $C(z,t)$ that is dimensionless and varies from -1 to $+1$ across the layer,

$$\overline{\rho}(z,t)=\rho_c[AC(z,t)+1], \tag{2.31}$$

where

$$\rho_c=\frac{1}{2}(\rho_1+\rho_2) \tag{2.32}$$

and A is the Atwood number

$$A=\frac{\rho_2-\rho_1}{\rho_2+\rho_1}=\frac{1}{2}\frac{\Delta_{\rho}}{\rho_c}. \tag{2.33}$$

The concentration function is dimensionless, and a self-similar solution would satisfy

$$C(z,t)=f_c(\chi), \tag{2.34}$$

with no explicit time dependence. The similarity function may be assumed to vary from -1 to $+1$. The self-similar forms of the density and concentration are related:

$$f_{\rho}(\chi)=\rho_c[Af_c(\chi)+1]. \tag{2.35}$$

From the fact that $\beta_0=0$, one can infer that single-point moments of the fluctuating density will yield similar results, e.g.,

$$\overline{\rho'^{n}}(z,t)=f_{\rho',n}(\chi). \tag{2.36}$$

These results apply to both incompressible KH and RT mixing layers and are a consequence of the assertion of self-similarity and the nature of the kinetic energy dissipation mechanism. No particular model or theory has been invoked.

C. Scaling of velocity and kinetic energy in the mixing layer

Since the RT mixing layer (and in some circumstances the KH layer) possesses fluids of different densities, the mass-weighted averages seem most appropriate for describing the energy of the flow. The mass-weighted average velocity is

$$\tilde{U}_i = \frac{\overline{\rho u_i}}{\overline{\rho}}, \tag{2.37}$$

where the overbar again denotes a planar average. Note that we have chosen a single-field representation of the velocity, and indices denote components of the vector. (A multifield velocity description could be analyzed as well, but since we are concerned only with the dimensionality of the functions, the results would be essentially the same.) The fluctuation about the average is

$$u_i'' = \tilde{U}_i - u_i. \tag{2.38}$$

If the density is constant across the layer (for example, in the KH mixing layer) the mass-weighted description is equivalent to the simple planar-averaged description. For the incompressible RT mixing layer the (planar-averaged) mean velocity is zero. This is because the net *volumetric* flux of heavy fluid into light must equal the net *volumetric* flux of light fluid into heavy to satisfy the ansatz of incompressibility. These decompositions are not restricted to a ‘‘turbulent’’ flow state—they may also be employed without ambiguity for stochastic flows at low Reynolds number.

The similarity analysis indicates that the characteristic speed $\mathcal{U}(z,t)$ for the KH or RT mixing layer has the self-similar form

$$\mathcal{U}(z,t) = U(t) f_u(\chi), \tag{2.39}$$

where

$$U(t) = U_0 \left[\frac{t+t_0}{t_0}\right]^{\gamma-1}, \tag{2.40}$$

independent of whether they are mass-weighted averages or simple planar averages.

For the KH mixing layer, $\gamma=1$ implies a self-similar form with an amplitude $U(t)$ that is constant in time. The function $f_u(\chi)$ is dimensionless and varies from -1 to $+1$ across the layer. The velocity difference across the mixing layer is $\Delta_U = U^+ - U^-$. Assume a frame wherein $U^+ = -U^- = U_{fs}$, where U_{fs} is the free-stream velocity. The mean velocity within the KH mixing layer is then

$$\overline{U_x}(z,t) = \frac{1}{2}\Delta_u f_u(\chi), \tag{2.41}$$

where Δ_U is again the difference of the free-stream velocities.

For the RT mixing layer, $\gamma=2$ gives linear growth in time of the characteristic velocity. If the characteristic velocity were assumed to be a simple planar-averaged velocity, then $U_0=0$ in the self-similar form. The function $f_u(\chi)$ is less than zero and approaches 0 as $z \to \pm\infty$. The normalization is taken that $f_u(0) = -1$ so that $U(t)$ is positive and equal to the absolute value of the mass-weighted velocity at the center of the mixing layer.

The total energy of fluctuations about the mass-weighted average is

$$\mathcal{Q}(z,t) = \frac{1}{2}\overline{\rho(\mathbf{x},t) u_n''(\mathbf{x},t) u_n''(\mathbf{x},t)}, \tag{2.42}$$

where a summation is implied over repeated indices. Self-similarity imposes the condition that

$$\tau^{\beta_0 + 2\gamma - 2}\mathcal{Q}(z,t) = \mathcal{Q}(\tau^\gamma z, \tau(t+t_0) - t_0). \tag{2.43}$$

However, the analysis of the density field indicated that $\beta_0 = 0$, from which one may infer that

$$\mathcal{Q}(z,t) = Q(t) f_Q(\chi), \tag{2.44}$$

where

$$Q(t) = Q_0 \left[\frac{t+t_0}{t_0}\right]^{2\gamma-2}. \tag{2.45}$$

The function $f_Q(\chi)$ has the normalization $f_Q(0)=1$, so that $\mathcal{Q}(0,t) = Q(t)$. This suggests that for the mass-weighted average characteristic velocity the time dependence of $U(t)$ may be conveniently rewritten as

$$U(t) = U_0 \left[\frac{Q(t)}{Q_0}\right]^{1/2}. \tag{2.46}$$

Such a form was also postulated by Sharp and co-workers [16]. The present result verifies that such a choice is a natural expression of the similarity of the flow.

The moments of the velocity fluctuations about the planar-averaged velocity field can be represented as follows. The specific fluctuational energy (fluctuational energy per unit mass)

$$\mathcal{K}(z,t) = \frac{1}{2}\overline{u_n'(\mathbf{x},t) u_n'(\mathbf{x},t)} \tag{2.47}$$

has the same scaling group as $Q(z,t)$ (so long as $\beta_0 = 0$), and the analysis yields

$$\mathcal{K}(z,t) = K(t) f_K(\chi), \tag{2.48}$$

where

$$K(t) = K_0 \left[\frac{t+t_0}{t_0}\right]^{2\gamma-2}, \tag{2.49}$$

and the function $f_K(\chi)$ has the normalization $f_K(0)=1$, so that $\mathcal{K}(0,t) = K(t)$. Now consider

$$\kappa_n(z,t) = \langle [u_l'(x,y,z,t) u_l'(x,y,z,t)]^{n/2} \rangle, \tag{2.50}$$

where $\mathcal{K}(z,t) = \kappa_2(t)$. The physical dimensionality of κ_n is $(L/T)^n$. The determining equation for κ_n is

$$\frac{n(\gamma-1)}{(t+t_0)}\kappa_n(z,t) = \frac{\partial \kappa_n(z,t)}{\partial z}\frac{\gamma z}{(t+t_0)} + \frac{\partial \kappa_n(z,t)}{\partial z}. \tag{2.51}$$

The solution is

$$\kappa_n(z,t)=K^{n/2}(t)f_{\kappa,n}(\chi) \tag{2.52}$$

with normalization

$$f_{\kappa,n}(0)=\frac{\kappa_n(0,t)}{K^{n/2}(t)} \tag{2.53}$$

at any time t. This result suggests that the skewness, hyperskewnesses, flatness, and hyperflatnesses of the velocity field become constant in time. This result is presented as a specific test of self-similarity that can be tested in simulations of RT and KH flows. (Our own computations suggest that this is violated at the edges of the mixing layer.)

The results presented in this subsection are consequences of the assumption of self-similarity and are independent of any assertions regarding the turbulent nature of the flow. Similar results have been implied in the analyses of two-phase flow models by Chen *et al.* [12], by Glimm, Saltz, and Sharp [18], and by others.

D. Energy dissipation rates in the mixing layer

The rate at which kinetic energy is dissipated to thermal fluctuations by the action of molecular viscosity may in many circumstances be strongly dependent on the specific nature of the flow and on the particular form of the viscous stress tensor. The rate of dissipation of energy $\mathcal{Q}(z,t)$ is denoted $\mathcal{E}(z,t)$ and has dimensions of $[(M/L^3)(L^2/T^3)]$, or, equivalently, density$\times[L^2/T^3]$. However, the dissipative terms in the underlying physical system (e.g., the Navier-Stokes equations) possesses an explicit dependence on viscosity ν, e.g.,

$$\mathcal{E}(z,t)=\nu\mathcal{E}_\nu(z,t), \tag{2.54}$$

where $\mathcal{E}_\nu(z,t)$ has dimensions of $[(M/L^3)(1/T^2)]$. Clearly, assuming that $\mathcal{E}_\nu$ is self-similar will lead to a different scaling than the assumption that $\mathcal{E}$ is self-similar. The latter assumption is consistent with the Kolmogorov notion of high-Reynolds-number, inertially driven turbulence wherein the dissipation rate is set by the energy cascade, rather than the specific details of the viscous dissipation and viscous scales. That is, by assuming that $\mathcal{E}$ is self-similar, we are making a tacit assumption that the dynamics of the turbulence are in some sense analogous to the more usual notions of turbulence. Physical consideration indicate that this ''turbulence-like'' assumption may be accurate in the core of the mixing layers where strong vorticity is generated and where the flow is not strongly intermittent. At the outside edges of the mixing layers this assumption is more questionable. We will make this assumption and determine the consequences for the self-similarity. We will also discuss the consequences of assuming that $\mathcal{E}_\nu(z,t)$ is self-similar.

The self-similar relationship for $\mathcal{E}(z,t)$ is

$$\tau^{\beta_0+2\gamma-3}\mathcal{E}(z,t)=\mathcal{E}(\tau^\gamma z,\tau(t+t_0)-t_0). \tag{2.55}$$

Again, we assert that $\beta_0=0$, and pursue an analysis equivalent to the analysis performed for $\overline{\rho}$ to yield

$$\mathcal{E}(z,t)=\epsilon(t)f_\epsilon(\chi), \tag{2.56}$$

where the dimensional, time-dependent function $\epsilon(t)$ may be written as

$$\epsilon(t)=\epsilon_0\left[\frac{t+t_0}{t_0}\right]^{2\gamma-3}=\epsilon_0\left[\frac{W_0}{Q_0^{3/2}}\right]\left[\frac{Q^{3/2}(t)}{W(t)}\right]=\frac{\zeta_0}{\rho_c^{1/2}}\frac{Q^{3/2}(t)}{W(t)}. \tag{2.57}$$

Note that the parameter ζ_0 is a constant. The factor $\rho_c^{1/2}$ is introduced to make ζ_0 dimensionless:

$$\zeta_0=\epsilon_0\left[\frac{W_0\rho_c^{1/2}}{Q_0^{3/2}}\right]. \tag{2.58}$$

This contrivance is awkward and might be avoided in various ways. For example, one might define the dissipation rate $\mathcal{Q}(z,t)$ as $\rho_c\tilde{\mathcal{E}}(z,t)$ where the function $\tilde{\mathcal{E}}$ has the dimensions of $[L^2/T^3]$. Likewise, one could represent the time dependence of $\epsilon(t)$ in terms of $K(t)$ instead of $Q(t)$ and introduce ρ_c instead of $\rho_c^{-1/2}$. However, the above form is preferred for the RT analysis, because it leads to a simpler form for the energy balance equation for the RT mixing layer.

The corresponding self-similar form for $\mathcal{E}_\nu(z,t)$ is

$$\mathcal{E}_\nu(z,t)=\epsilon_\nu(t)f_{\epsilon_\nu}(\chi), \tag{2.59}$$

where

$$\epsilon_\nu(t)=\epsilon_{\nu,0}\left[\frac{t+t_0}{t_0}\right]^{-2}=\epsilon_{\nu,0}\left[\frac{W_0}{W(t)}\right]. \tag{2.60}$$

From simple dimensional considerations the dissipative length scales for these flows have the form

$$\eta(z,t)=\left[\frac{\rho_c\nu_c^3}{\mathcal{E}(z,t)}\right]^{1/4}, \tag{2.61}$$

where ν_c is a measure of the kinematic viscosity of the two fluids. This length scale is analogous to the Kolmogorov dissipation scale associated with the more usual hydrodynamic turbulence. For the viscosity-independent scaling given by Eqs. (2.56)–(2.58), this length scale has a time dependence of $t^{-(2\gamma-3)/4}$ for the turbulencelike assumption and $t^{1/2}$ for the case of $\mathcal{E}_\nu$. The $t^{1/2}$ scaling is consistent with a viscous scaling [recall Eq. (2.18)], which should be expected since it was constructed from a definition of dissipation [Eq. (2.54)] that explicitly referenced the viscosity as a physically relevant parameter.

Another length scale analogous to the Taylor microscale may be constructed, e.g.,

$$\lambda(z,t)=\left[\nu_c\frac{\mathcal{Q}(z,t)}{\mathcal{E}(z,t)}\right]^{1/2}. \tag{2.62}$$

This length scale has a time dependence of t^{+1} for both the turbulencelike assumption and the case of $\mathcal{E}_\nu$.

Thus we see the emergence of three distinct length scales, a viscous scale η, analogous to a Kolmogorov dissipative scale, an intermediate, viscous-dynamical scale λ, analogous to the Taylor microscale, and a large integral scale $W(t)$. The emergence of these various additional length scales is consistent with self-similarity. Of them, the viscous scale growth rate depends on assumptions regarding the nature of the dissipative processes. The time exponents of the viscous-dynamical scale growth rate and of the large-scale growth rate are independent of particular assumptions regarding the dissipative scales.

III. ENERGY SOURCES

The KH and RT mixing layers have ''external'' sources of energy that are converted to kinetic energy and dissipated. For the Rayleigh-Taylor case, this source is the potential energy of the fluids. For the Kelvin-Helmholtz case, the energy source is the velocity difference across the mixing layer. Each case will be analyzed separately.

A. The Rayleigh-Taylor mixing layer

A precise definition of the potential energy for the RT mixing layer requires a precise definition of the position of the layer in the direction of the acceleration. A coordinate z is now introduced in the direction of the acceleration wherein $z=0$ corresponds to the center of the mixing layer. For purposes of the present calculation we will consider the total potential energy over a domain in z that extends from a point $z=-z_{\mathrm{ref}}$ below the lower edge $z=-W(t)/2$ to a point $z=+z_{\mathrm{ref}}$ that is above the mixing-layer edge $z=+W(t)/2$. For simplicity we have not distinguished between the bubble-side and spike-side dynamics—such a distinction appears to require a specific model or theory of the RT mixing layer. The potential energy then becomes

$$\Pi(t)=\int_{-z_{\mathrm{ref}}}^{+z_{\mathrm{ref}}}\overline{\rho(z)}g(z+z_{\mathrm{ref}})dz=\int_{-z_{\mathrm{ref}}}^{+z_{\mathrm{ref}}}\overline{\rho(z)}gz\,dz+gz_{\mathrm{ref}}\int_{-z_{\mathrm{ref}}}^{+z_{\mathrm{ref}}}\overline{\rho(z)}dz. \tag{3.1}$$

The planar-averaged density integrated over the volume $-z_{\mathrm{ref}}\leqslant z\leqslant z_{\mathrm{ref}}$ is

$$\overline{\rho_\Sigma}=\int_{-z_{\mathrm{ref}}}^{+z_{\mathrm{ref}}}\overline{\rho(z)}dz, \tag{3.2}$$

so that

$$\Pi(t)=\int_{-z_{\mathrm{ref}}}^{+z_{\mathrm{ref}}}\overline{\rho(z)}g(z+z_{\mathrm{ref}})dz=g\int_{-z_{\mathrm{ref}}}^{+z_{\mathrm{ref}}}\overline{\rho(z)}z\,dz+gz_{\mathrm{ref}}\overline{\rho_\Sigma}. \tag{3.3}$$

The summed density may be written as

$$\begin{aligned}\overline{\rho_\Sigma}&=\int_{-z_{\mathrm{ref}}}^{-W(t)/2}\rho_1dz+\int_{-W(t)/2}^{+W(t)/2}\rho_c[Ac(z,t)+1]dz+\int_{+W(t)/2}^{+z_{\mathrm{ref}}}\rho_2\\&=\rho_1\int_{-z_{\mathrm{ref}}}^{-W(t)/2}dz+\rho_c\int_{-W(t)/2}^{+W(t)/2}dz+\rho_2\int_{+W(t)/2}^{+z_{\mathrm{ref}}}dz+\rho_cA\int_{-W(t)/2}^{+W(t)/2}C(z,t)dz\\&=(\rho_1z_{\mathrm{ref}}+\rho_2z_{\mathrm{ref}})+\rho_cAW(t)I_{c,0},\end{aligned} \tag{3.4}$$

where $I_{c,n}$ is the nth moment of $f_c(\chi)$,

$$I_{c,n}=\int_{-1/2}^{+1/2}f_c(\chi)\chi^n\,d\chi. \tag{3.5}$$

If the fluids are incompressible and there is no net mass flux in the z direction at $z=\pm z_{\mathrm{ref}}$, then the total mass in the volume in the range $-z_{\mathrm{ref}}\leqslant z\leqslant z_{\mathrm{ref}}$ is constant in time. This is equivalent to a solid boundary at $z=\pm z_{\mathrm{ref}}$, though we can consider the circumstance wherein the walls are at $z_{\mathrm{ref}}\rightarrow\infty$. This zero-flux condition requires that the zeroth moment of $f_c(\chi)$, $I_{c,0}=0$, yielding

$$\overline{\rho_\Sigma}=(\rho_1+\rho_2)z_{\mathrm{ref}}=2\rho_cz_{\mathrm{ref}}. \tag{3.6}$$

The doubly averaged density (i.e., the planar-averaged density averaged over $-z_{\mathrm{ref}}\leqslant z\leqslant z_{\mathrm{ref}}$) is

$$\langle\overline{\rho}\rangle=\frac{\overline{\rho_\Sigma}}{2z_{\mathrm{ref}}}=\rho_c. \tag{3.7}$$

Now consider the integral

$$\Pi_1(t)=\Pi(t)-gz_{\rm ref}\overline{\rho_\Sigma}=g\int_{-z_{\rm ref}}^{+z_{\rm ref}}\overline{\rho}(z)z\,dz, \tag{3.8}$$

or, equivalently,

$$\begin{aligned}\frac{1}{g}\Pi_1(t)&=\int_{-z_{\rm ref}}^{-W(t)/2}\rho_1 z\,dz+\int_{-W(t)/2}^{+W(t)/2}\rho_c[AC(z,t)+1]z\,dz+\int_{+W(t)/2}^{+z_{\rm ref}}\rho_2 z\,dz\\ &=\rho_1\int_{-z_{\rm ref}}^{-W(t)/2}z\,dz+\rho_c\int_{-W(t)/2}^{+W(t)/2}z\,dz+\rho_2\int_{+W(t)/2}^{+z_{\rm ref}}z\,dz+\rho_c A\int_{-W(t)/2}^{+W(t)/2}C(z,t)z\,dz\\ &=\frac{1}{2}(\rho_2-\rho_1)z_{\rm ref}^2+\rho_c AW^2(t)\left[I_{c,1}-\frac{1}{4}\right].\end{aligned} \tag{3.9}$$

Thus the total potential energy is

$$\begin{aligned}\Pi(t)&=gz_{\rm ref}\overline{\rho_\Sigma}+\frac{g}{2}(\rho_2-\rho_1)z_{\rm ref}^2+\rho_c gAW^2(t)\left[I_{c,1}-\frac{1}{4}\right]\\ &=\rho_c g\{2z_{\rm ref}^2+Az_{\rm ref}^2-AW^2(t)\tilde{I}_{c,1}\},\end{aligned} \tag{3.10}$$

where we have let

$$\tilde{I}_{c,1}=\left[\frac{1}{4}-I_{c,1}\right]. \tag{3.11}$$

The total potential energy is dependent on the reference position $z_{\rm ref}$. However, the rate of change of potential energy is not

$$\frac{d\Pi(t)}{dt}=-\rho_c gA\tilde{I}_{c,1}2W(t)\frac{dW(t)}{dt}. \tag{3.12}$$

Determination of the value of the moment $I_{c,1}$ requires a specific model or theory of the RT flow. Such model-dependent details are beyond the scope of the present paper, which merely seeks to explain the origin of the self-similar forms without resort to specific models or theories.

B. The Kelvin-Helmholtz mixing layer

The energy source of the KH mixing layer is the mean flow, which has infinite spatial extent. Recall that the free-stream velocities are U_1 and U_2 and the velocity difference across the layer is given by Eq. (2.14). We will define the energy relative to a frame moving at the average velocity, i.e., we are in the frame where $\Sigma_u=U_2+U_1=0$. Thus $U_2=+\Delta_u/2$ and $U_1=-\Delta_u/2$. The total spatially integrated mean flow kinetic energy $\mathcal{M}_T(t)$ may be constructed about some reference volume over the domain $-z_{\rm ref}\leqslant z\leqslant +z_{\rm ref}$ which again encompasses the shear layer $-W(t)/2\leqslant z\leqslant +W(t)/2$ for any time of interest:

$$\begin{aligned}\mathcal{M}_T(t)&=\int_{-z_{\rm ref}}^{+z_{\rm ref}}\overline{\rho}(z,t)\overline{U_x}^2(z,t)dz\\ &=\int_{-z_{\rm ref}}^{-W(t)/2}\rho_1\left(-\frac{1}{2}\Delta_u\right)^2dz\\ &\quad+\int_{-W(t)/2}^{+W(t)/2}\overline{\rho}(z,t)\overline{U_x}^2(z,t)dz\\ &\quad+\int_{W(t)/2}^{+z_{\rm ref}}\rho_2\left(\frac{1}{2}\Delta_u\right)^2dz,\end{aligned} \tag{3.13}$$

or, exploiting the self-similarity and letting $\chi_{\rm ref}=z_{\rm ref}/W(t)$,

$$\begin{aligned}\mathcal{M}_T(t)&=\left(\frac{1}{2}\Delta_u\right)^2W(t)\left\{[\rho_1+\rho_2]\left(\chi_{\rm ref}-\frac{1}{2}\right)\right.\\ &\quad\left.+\rho_c\int_{-1/2}^{+1/2}[Af_c(\chi)+1]f_u^2(\chi)d\chi\right\}.\end{aligned} \tag{3.14}$$

This reduces to

$$\begin{aligned}\mathcal{M}_T(t)&=\rho_c\left(\frac{1}{2}\Delta_u\right)^2W(t)\{[2\chi_{\rm ref}-1]+AI_{cu^2}+I_{u^2}\}\\ &=\rho_c\left(\frac{1}{2}\Delta_u\right)^2W(t)\left\{\left[2\frac{z_{\rm ref}}{W(t)}-1\right]+AI_{cu^2}+I_{u^2}\right\}\\ &=\rho_c\left(\frac{1}{2}\Delta_u\right)^2\{2z_{\rm ref}-\tilde{I}_{u^2}W(t)\},\end{aligned} \tag{3.15}$$

where $\tilde{I}_{u^2}$ is assumed to be positive:

$$\tilde{I}_{u^2}=[1-AI_{cu^2}-I_{u^2}]=1-\int_{-1/2}^{+1/2}[Af_c(\chi)+1]f_u^2(\chi)d\chi. \tag{3.16}$$

As in the case of the Rayleigh-Taylor mixing-layer potential energy, this definition of source energy gives infinity as $z_{\rm ref}\to\infty$. Again, the rate of change of the source energy is bounded and independent of the choice of $z_{\rm ref}$:

$$\frac{d\mathcal{M}_T(t)}{dt} = -\rho_c \tilde{I}_{u^2}\left(\frac{1}{2}\Delta_u\right)^2 \frac{dW(t)}{dt} = -\rho_c \tilde{I}_{u^2} U_{f_s}^2 \frac{dW(t)}{dt}. \tag{3.17}$$

Again, the moment $\tilde{I}_{u^2}$ must be determined from a particular model or theory, which is beyond the scope and intent of this paper.

IV. ENERGY BALANCES

Now we note that the energy balance for the mixing layers is

$$\frac{d}{dt}\{\text{total energy}\} = \{\text{dissipation}\},$$

$$\frac{d}{dt}\{\Pi(t) + \mathcal{M}_T(t) + \mathcal{Q}_T(t)\} = -\mathcal{E}_T(t), \tag{4.1}$$

where $\Pi(t)$ is the total potential energy, $\mathcal{M}_T(t)$ is the total mean flow energy, $\mathcal{Q}_T(t)$ is the total fluctuational energy, and $\mathcal{E}_T(t)$ is the total dissipation rate. The word ''total'' is again taken to mean ''integrated over the domain.'' The velocities will be constructed as mass-weighted (i.e., Favre) averages. Thus there will be a mean flow in the case of the RT mixing layer.

A. The energy balance for the Rayleigh-Taylor mixing layer

The kinetic energy integrated over all space $\mathcal{Q}_T(t)$ in the RT system is given by

$$\mathcal{Q}_T(t) = \int_{-\infty}^{+\infty} \mathcal{Q}(z,t)\,dz$$

$$= Q(t)\int_{-\infty}^{+\infty} f_Q(\chi)\,d(W(t)\chi)$$

$$= Q(t)W(t)I_Q, \tag{4.2}$$

where

$$I_Q = \int_{-\infty}^{+\infty} f_Q(\chi)\,d\chi. \tag{4.3}$$

The dissipation rate integrated over the entire layer is

$$\mathcal{E}_T(t) = \int_{-\infty}^{+\infty} \mathcal{E}(z,t)\,dz$$

$$= \epsilon(t)\int_{-\infty}^{+\infty} f_\epsilon(\chi)\,d(W(t)\chi)$$

$$= \epsilon(t)W(t)I_\epsilon = \frac{\zeta_0 I_\epsilon}{\rho_c^{1/2}} Q^{3/2}(t), \tag{4.4}$$

where

$$I_\epsilon = \int_{-\infty}^{+\infty} f_\epsilon(\chi)\,d\chi. \tag{4.5}$$

The mean kinetic energy integrated over the entire layer is

$$\mathcal{M}_T(t) = \int_{-\infty}^{+\infty} \overline{\rho(z,t)}\,\tilde{U}^2(z,t)\,dz$$

$$= \rho_c \frac{U_0^2}{Q_0} Q(t)\int_{-\infty}^{+\infty} [Af_c(\chi)+1] f_u^2(\chi)\,d(W(t)\chi)$$

$$= \rho_c \frac{U_0^2}{Q_0} Q(t)W(t)(AI_{cu^2} + I_{u^2})$$

$$= \frac{U_0^2}{\mathcal{U}_Q^2} Q(t)W(t)(AI_{cu^2} + I_{u^2}), \tag{4.6}$$

where

$$I_{cu^2} = \int_{-\infty}^{+\infty} f_c(\chi) f_u^2(\chi)\,d\chi, \tag{4.7}$$

$$I_{u^2} = \int_{-\infty}^{+\infty} f_u^2(\chi)\,d\chi, \tag{4.8}$$

and

$$\mathcal{U}_Q = \left[\frac{Q_0}{\rho_c}\right]^{1/2}. \tag{4.9}$$

The total energy balance may be written as

$$-2\tilde{I}_{c,1}\rho_c A g W(t)\frac{dW(t)}{dt} + \left[Q(t)\frac{dW(t)}{dt} + W(t)\frac{dQ(t)}{dt}\right] \times \left[I_Q + \left(\frac{U_0}{\mathcal{U}_Q}\right)^2 (AI_{cu^2} + I_{u^2})\right] = -\frac{\zeta_0 I_\epsilon}{\rho_c^{1/2}} Q^{3/2}(t), \tag{4.10}$$

or, equivalently,

$$-2\rho_c A g \frac{dW(t)}{dt} + J_Q\left[\frac{Q(t)}{W(t)}\frac{dW(t)}{dt} + \frac{dQ(t)}{dt}\right] = -\frac{J_\epsilon}{\rho_c^{1/2}}\frac{Q^{3/2}(t)}{W(t)}, \tag{4.11}$$

where

$$J_Q = \frac{1}{\tilde{I}_{c,1}}\left[I_Q + \left(\frac{U_0}{\mathcal{U}_Q}\right)^2 (AI_{cu^2} + I_{u^2})\right], \tag{4.12}$$

$$J_\epsilon = \zeta_0 \frac{I_\epsilon}{\tilde{I}_{c,1}}. \tag{4.13}$$

The energy balance for self-similarity is

$$-4Ag\frac{W_0}{t_0}\left(\frac{t+t_0}{t_0}\right) + 4J_Q\frac{Q_0}{\rho_c t_0}\left(\frac{t+t_0}{t_0}\right) = -J_\epsilon \frac{Q_0^{3/2}}{\rho_c^{3/2} W_0}\left(\frac{t+t_0}{t_0}\right), \tag{4.14}$$

which simplifies to

$$AgW_0^2 - J_Q\mathcal{U}_Q^2 W_0 - \frac{J_\epsilon}{4} t_0 \mathcal{U}_Q^3 = 0. \tag{4.15}$$

Solving for W_0 yields

$$W_0 = \frac{1}{2} J_Q \frac{\mathcal{U}_Q^2}{Ag}\left\{ 1 \pm \left[1 + \frac{J_\epsilon}{J_Q^2} \frac{Agt_0}{\mathcal{U}_Q} \right]^{1/2} \right\}. \tag{4.16}$$

The above equation suggests that $W_0 \propto [Ag]^{-1}$. This seems unreasonable—if A or g vanishes, then W_0 should vanish. However, the moments (J_Q, J_ϵ) and virtual origin data (Q_0, t_0) may also be functions of the Atwood number. This suggests that $\mathcal{U}_Q \propto [Ag]$. Accordingly, we introduce a dimensionless parameter G_0 satisfying

$$\mathcal{U}_Q = 2G_0 t_0 Ag,$$

so that

$$W_0 = 2Ag(G_0^2 J_Q t_0^2)\left\{ 1 \pm \left[1 + \frac{1}{2} \frac{J_\epsilon}{G_0 J_Q^2} \right]^{1/2} \right\}. \tag{4.17}$$

The usual assumption regarding the RT mixing layer is that the bubbles (lighter fluid penetrating heavy fluid) grow as

$$h_B(t) = \alpha_B(A) A g t^2, \tag{4.18}$$

and spikes grow as

$$h_S(t) = \alpha_B(A) A g t^2, \tag{4.19}$$

where α_B and α_S are functions of the Atwood number. The width of the mixing layer then becomes $W(t) = h_B(t) + h_S(t)$,

$$W(t) = [\alpha_B(A) + \alpha_S(A)]Agt^2 = 2\alpha(A)Agt^2, \tag{4.20}$$

where

$$\alpha(A) = \frac{1}{2}[\alpha_B(A) + \alpha_S(A)]. \tag{4.21}$$

Comparing this to the self-similar form for $W(t)$ in the limit of $t \to \infty$ allows the identification

$$W_0 = 2\alpha(A) A g t_0^2, \tag{4.22}$$

for the case of $t \gg t_0$. The results of the analysis of the growth rate now may be recast in terms of α:

$$\alpha(A) = \frac{W_0}{2Agt_0^2} = J_Q G_0^2 \left\{ 1 \pm \left[1 + \frac{1}{2} \frac{J_\epsilon}{G_0 J_Q^2} \right]^{1/2} \right\}. \tag{4.23}$$

It is reasonable to assume that $\alpha(A) > 0$, i.e., that the mixing layer grows in time. In addition, we have assumed that $t_0 > 0$ [if t_0 had been negative, then the entire analysis would be rewritten using $t - t_0/(-t_0)$ and the equation above would involve $-t_0$—requiring that $-t_0 > 0$ and thus leading to the same conclusions]. These two assumptions imply $G_0 > 0$. In addition,

$$J_Q > 0 \tag{4.24}$$

and

$$J_\epsilon > 0. \tag{4.25}$$

Thus we can identify the particular root needed as

$$\alpha(A) = J_Q G_0^2 \left\{ 1 + \left[1 + \frac{1}{2} \frac{J_\epsilon}{G_0 J_Q^2} \right]^{1/2} \right\}. \tag{4.26}$$

This equation represents the functional form of the RT growth-rate parameter α. If α is to be a universal constant at a given Atwood number, the right side of this equation must be independent of initial conditions—either parameter by parameter or collectively. Unfortunately, this self-similar analysis does not indicate the values of any of the parameters—these must be deduced from a specific theory, model, or physical system (i.e., an experiment). Values of α_B determined experimentally are in the range of 0.05 to 0.07 at modest Atwood number ($A \ll 1$) and approach 0.5 as the Atwood number approaches 1. The α_S are typically larger than α_B and their ratio is dependent on Atwood number [19–23]. The results shown in Eq. (4.26) does rely on one crucial assumption regarding turbulence—that the dissipation rate is independent of the value of the viscosity and its self-similar form is given by Eqs. (2.56)–(2.58). Had the form given by Eqs. (2.59) and (2.60) been used, the value for α would be modified to

$$\alpha(A) = 2J_Q G_0^2. \tag{4.27}$$

B. The energy balance for the Kelvin-Helmholtz mixing layer

For the KH shear flow, the source of energy is the mean flow and the potential energy is zero. The kinetic energy and dissipation rates integrated over the entire space have the same form as that for the RT mixing layer. The energy balance is thus

$$-\rho_c \tilde{I}_{u^2} U_{fs}^2 \frac{dW(t)}{dt} + \left[Q(t)\frac{dW(t)}{dt} + W(t)\frac{dQ(t)}{dt} \right] I_Q = -\frac{\zeta_0 I_\epsilon}{\rho_C^{1/2}} Q^{3/2}(t). \tag{4.28}$$

Rewriting this, we have

$$-U_{fs}^2 \frac{dW(t)}{dt} + \frac{J_Q}{\rho_c}\left[Q(t)\frac{dW(t)}{dt} + W(t)\frac{dQ(t)}{dt} \right] = -\frac{J_\epsilon}{\rho_c^{3/2}} Q^{3/2}(t), \tag{4.29}$$

where

$$J_Q=\frac{I_Q}{\tilde{I}_{u^2}}, \quad (4.30)$$

$$J_\epsilon=\frac{I_\epsilon}{\tilde{I}_{u^2}}. \quad (4.31)$$

The energy balance for the self-similarity is

$$\frac{W_0}{t_0}\left[J_\epsilon\frac{Q_0}{\rho_c}-U_{fs}^2\right]=-J_Q\left[\frac{Q_0}{\rho_c}\right]^{3/2} \quad (4.32)$$

or

$$\frac{W_0}{t_0}=\frac{Q_0}{\rho_c U_{fs}^2}\left[\frac{Q_0}{\rho_c}\right]^{1/2}\frac{J_Q}{\{1-J_\epsilon(Q_0/\rho_c U_{fs}^2)\}}. \quad (4.33)$$

Now let

$$R_0=\frac{Q_0}{\rho_c U_{fs}^2}, \quad (4.34)$$

so

$$\frac{W_0}{t_0}=\frac{J_Q R_0^{3/2}}{[1-J_\epsilon R_0]}U_{fs}=\frac{J_Q R_0^{3/2}}{2[1-J_\epsilon R_0]}\Delta_u, \quad (4.35)$$

suggesting that

$$W(t)=\frac{J_Q R_0^{3/2}}{[1-J_\epsilon R_0]}U_{fs}(t+t_0)=\frac{J_Q R_0^{3/2}}{2[1-J_\epsilon R_0]}\Delta_u(t+t_0). \quad (4.36)$$

Alternatively, if the dissipation were assumed to be of the form given by Eqs. (2.59) and (2.60), then the above relationship would be modified to

$$W(t)=J_Q R_0^{3/2}U_{fs}(t+t_0)=\frac{J_Q R_0^{3/2}}{2}\Delta_u(t+t_0). \quad (4.37)$$

One may infer from these equations that $J_\epsilon R_0<1$. As is the case with the RT mixing-layer growth parameter, the specific values of the parameters in this expression must be determined from a specific model, theory, or experiment. Independence from initial conditions again implies that the various parameters in this expression must be independent of initial conditions either parameter by parameter or collectively. Note that experimental evidence suggests that W_0/t_0 is approximately 0.14–0.22 [24,25].

V. SELF-SIMILARITY OF INHOMOGENEOUS SPECTRA

A. Scaling of energy spectra

The spectral representation of an inhomogeneous function is, to some extent, arbitrary. For the case of the RT mixing layer and the KH shear layer, several choices are obvious. For convenience, we will assume that the directions perpendicular to the acceleration or shear gradient (i.e., perpendicular to the z axis) are statistically homogeneous and thus may be aptly represented in terms of Fourier integral transforms. For the inhomogeneous direction z, one choice is to expand in the inhomogeneous directions using appropriate basis functions (e.g., Hermite functions for bounded systems, and trigonometric functions or Jacobi polynomials for bounded domains). Another possibility is to assume a representation that describes the spectra of the correlations in terms of Fourier integral transforms of the two-point correlations in all three directions at each point in the inhomogeneous domain. Besnard *et al.* [26] used an inhomogeneous spectrum based on a Wigner representation of the inhomogeneous spectrum to produce a spectrum of the form $E(\mathbf{x},k,t)$ where $k^2=k_x^2+k_y^2+k_z^2$. For the case of a single inhomogeneous direction z, the form becomes $E(z,k,t)$. Zhou [27] defined a spectrum for a mixing layer of the form $E(z,k,t)$ where z is the inhomogeneous direction perpendicular to the plane of the mixing layer and $k^2=k_x^2+k_y^2$. Note that these spectral definitions (that of Zhou and of Besnard *et al.*) share the same fundamental dimensionality, as do their arguments, and thus share the same self-similar functional forms.

The minimal assumption required to begin the analysis is that the energy spectrum $E(z,k,t)$ has the dimensions of $[L^3T^{-2}]$ so that upon integration over all wave numbers the result is a turbulent kinetic energy $q(z,t)$ with dimensions $[L^2T^{-2}]$. We will also seek a spectral representation for the turbulent kinetic energy dissipation rate $\mathcal{D}(z,k,t)$. The dimensions are $[L^3T^{-3}]$. The rationale for this representation of the dissipation rate is that viscosity ($\gamma=1/2$) does not scale by the same group as the velocity ($\gamma=1$) or acceleration ($\gamma=2$). Thus we seek a representation that is independent of viscosity. Thus we assume a cascadelike process wherein the energy is carried to small scales where it is eventually dissipated by viscosity at a rate and in a manner that is independent of the actual numerical value of the viscosity. This is the picture that Kolmogorov describes for isotropic turbulence—we adopt it here out of convenience and in the hope that it is not an inaccurate conjecture for these flows.

We begin by noting that the simple form of the spectrum describing isotropic turbulence is not appropriate for a strongly inhomogeneous system such as a mixing layer. We will utilize the subgroup consistent with the analysis for $W(t)$. Now we have

$$\tau^{(3\gamma-2)}E(z,k,t)=E(\tau^\gamma z,\tau^{-\gamma}k,\tau(t+t_0)-t_0). \quad (5.1)$$

Differentiating with respect to τ gives

$$(3\gamma-2)\tau^{(3\gamma-3)}E(z,k,t)=\frac{\partial E(z,k,t)}{\partial z}\gamma z\tau^{(\gamma-1)}-\frac{\partial E(z,k,t)}{\partial k}\gamma k\tau^{-(\gamma+1)}+\frac{\partial E(z,k,t)}{\partial t}(t+t_0). \quad (5.2)$$

Setting $\tau=1$ gives the determining equation

$$(3\gamma-2)E(z,k,t)=\frac{\partial E(z,k,t)}{\partial z}\gamma z-\frac{\partial E(z,k,t)}{\partial k}\gamma k+\frac{\partial E(z,k,t)}{\partial t}(t+t_0). \quad (5.3)$$

For convenience, we divide through by $(t+t_0)$:

$$\frac{(3\gamma-2)}{(t+t_0)}E(z,k,t)=\frac{\partial E(z,k,t)}{\partial z}\frac{\gamma z}{(t+t_0)}-\frac{\partial E(z,k,t)}{\partial k}\frac{\gamma k}{(t+t_0)}+\frac{\partial E(z,k,t)}{\partial t}. \quad (5.4)$$

Characteristics in k are

$$\frac{dk(t)}{dt}=-\frac{\gamma k(t)}{(t+t_0)}, \quad (5.5)$$

giving

$$k(t)=k_0\left[\frac{t+t_0}{t_0}\right]^{-\gamma}. \quad (5.6)$$

Characteristics in z are

$$\frac{dz(t)}{dt}=\frac{\gamma z(t)}{(t+t_0)}, \quad (5.7)$$

giving

$$z(t)=z_0\left[\frac{t+t_0}{t_0}\right]^{\gamma}. \quad (5.8)$$

Now, we divide the determining equation for E by $(t+t_0)$, and substituting the characteristic equations yields

$$\frac{(3\gamma-2)}{(t+t_0)}E(z,k,t)=\frac{\partial E(z,k,t)}{\partial z}\frac{dz(t)}{dt}+\frac{\partial E(z,k,t)}{\partial k}\frac{dk(t)}{dt}+\frac{\partial E(z,k,t)}{\partial t}=\frac{dE(z(t),k(t),t)}{dt}. \quad (5.9)$$

Solving this along the characteristic gives

$$E(z(t),k(t),t)=E_0(z_0,k_0)\left[\frac{t+t_0}{t_0}\right]^{(3\gamma-2)}. \quad (5.10)$$

Now apply the characteristic solutions for $k(t)$ and $z(t)$:

$$E(z(t),k(t),t)=E_0\left(z(t)\left[\frac{t+t_0}{t_0}\right]^{-\gamma},k(t)\left[\frac{t+t_0}{t_0}\right]^{\gamma}\right)\times\left[\frac{t+t_0}{t_0}\right]^{(3\gamma-2)}. \quad (5.11)$$

Now define

$$W(t)=W_0\left[\frac{t+t_0}{t_0}\right]^{\gamma} \quad (5.12)$$

and

$$K(t)=K_0\left[\frac{t+t_0}{t_0}\right]^{2\gamma-2}. \quad (5.13)$$

Note that the expression for $K(t)$ is precisely the result that would be produced by applying the same principles to $K(T)$ that were applied to $W(t)$ and $E(z,k,t)$, demonstrating the consistency of the approach. The self-similar form of $E(z,k,t)$ may now be rewritten as

$$E(z,k,t)=K(t)W(t)f(\chi,\xi), \quad (5.14)$$

where

$$\chi=\frac{z}{W(t)}, \quad (5.15)$$

$$\xi=kW(t), \quad (5.16)$$

and

$$f(\chi,\xi)=(K_0W_0)^{-1}E_0(\chi W_0,\xi/L_0). \quad (5.17)$$

We now will constrain $f(\chi,\xi)$ as follows:

$$\int_0^\infty f(0,\xi)d\xi=1, \quad (5.18)$$

so that

$$\int_0^\infty E(0,k,t)dk=K(t)\int_0^\infty f(0,\xi)d\xi=K(t), \quad (5.19)$$

that is, $K(t)$ is the kinetic energy of the mixing layer at the midplane $z=0$.

The physical acceleration g implies the same constraint on γ as shown for the model of $W(t)$ in the previous section. For the RT mixing layer ($\gamma=2$) $W(t)$ and $K(t)$ grow as t^2. For the KH mixing layer ($\gamma=1$) W grows as t and K behaves as t^0.

Now define the turbulent kinetic energy integrated across the mixing layer as

$$\mathcal{Q}_T(t)=\int_{-\infty}^{+\infty}\int_0^\infty E(z,k,t)dk\,dz=K(t)W(t)I_f=K_0\frac{Q(t)}{Q_0}W(t)I_f, \quad (5.20)$$

where

$$I_f=\int_{-\infty}^{\infty}\int_0^\infty f(\chi,\xi)d\xi\,d\chi. \quad (5.21)$$

If it is assumed that the two-point correlation functions of the velocity field possess algebraic tails at large separations, then the infrared (low wave number) portion of the spectrum may be presumed to be a power law in k:

$$\lim_{k\to 0} E(z,k,t)=E_0(z,t)k^n, \tag{5.22}$$

where E_0 has dimensions of $L^{3+n}T^2$. Then we may write

$$\tau(3+n)\gamma-2E_0(x,t)=E_0(\tau^\gamma z,\tau(t+t_0)-t_0). \tag{5.23}$$

Solving this in the same manner as was done for $E(z,k,t)$ yields

$$E_0(x,t)=e_0\left[\frac{t+t_0}{t_0}\right]^{(3+n)\gamma-2} f_{E_0}(\chi). \tag{5.24}$$

Note that for simple (isotropic or anisotropic) homogeneous turbulence the large scales are assumed to be invariant [28]. This assumption implies that $(3+n)\gamma-2=0$. For decaying homogeneous turbulence this relationship represents a constraint on γ and n and was exploited by Clark and Zemach to determine the decay laws of isotropic turbulence. For the case of mixing layers, the only possible infrared scaling that is consistent with both the permanence of large scales and the accelerative scaling $\gamma=2$ is $n=-2$. The case of $n=-2$ was identified previously by Inogamov [29,30] and more recently in work by Dimonte [31–34]. However, the value of $n=-2$ yields spectra that are divergent at small wavenumbers, and thus may indicate that the largest scales are not invariant, but grow with the mixing layer in a prescribed manner dependent on the scaling exponent n.

B. Inhomogeneous dissipation rate $\mathcal{D}(z,k,t)$

We assume that an energy transfer spectrum $\mathcal{T}(z,k,t)$ (i.e., a function that transfers energy to some dissipative ''range'') has dimensions of L^3T^{-3}, giving

$$\tau^{(3\gamma-3)}T\mathcal{T}(z,k,t)=\mathcal{T}(\tau^\gamma z,\tau^{-\gamma}k,\tau(t+t_0)-t_0). \tag{5.25}$$

Following the same procedure as established for $E(z,k,t)$,

$$\frac{(3\gamma-3)}{(t+t_0)}\mathcal{T}(z,k,t)=\frac{\partial\mathcal{T}(z,k,t)}{\partial z}\frac{\gamma z}{(t+t_0)}-\frac{\partial\mathcal{T}(z,k,t)}{\partial k}\frac{\gamma k}{(t+t_0)}+\frac{\partial\mathcal{T}(z,k,t)}{\partial t}. \tag{5.26}$$

The characteristics are unchanged, yielding a solution along the characteristic of the form

$$\mathcal{T}(z(t),k(t),t)=\mathcal{T}_0(z_0,k_0)\left[\frac{t+t_0}{t_0}\right]^{(3\gamma-3)}. \tag{5.27}$$

Now apply the characteristic solutions for $k(t)$ and $z(t)$:

$$\mathcal{T}(z(t),k(t),t)=\mathcal{T}_0\left(z(t)\left[\frac{t+t_0}{t_0}\right]^{-\gamma},k(t)\left[\frac{t+t_0}{t_0}\right]^{\gamma}\right)\times\left[\frac{t+t_0}{t_0}\right]^{(3\gamma-3)}. \tag{5.28}$$

Then the self-similar form can be rewritten as

$$\mathcal{T}(z,k,t)=K^{3/2}(t)f_{\mathcal{T}}(\chi,\xi). \tag{5.29}$$

We will identify the dissipation $\epsilon(z,t)$ with the energy transfer:

$$\epsilon(z,t)=\int_0^\infty\mathcal{T}(z,k,t)dk=\frac{K^{3/2}(t)}{W(t)}\int_0^\infty f_{\mathcal{T}}(\chi,\xi)d\xi. \tag{5.30}$$

Note that the energy transfer is presumed to be integrable on this interval. In practical circumstances, we may envision a large-wave-number region where the transfer deposits energy in viscous dissipation. In this circumstance, the transfer vanishes at high k and the integral is finite. For convenience, we define

$$\epsilon(0,t)=\int_0^\infty\mathcal{T}(0,k,t)dk=\frac{K^{3/2}(t)}{W(t)}\int_0^\infty f_{\mathcal{T}}(0,\xi)d\xi=\epsilon_0\frac{K^{3/2}(t)}{W(t)}, \tag{5.31}$$

or

$$\epsilon_0=\int_0^\infty f_{\mathcal{T}}(0,\xi)d\xi. \tag{5.32}$$

For the case of vanishing viscosity (i.e., very high Reynolds number) we anticipate that remains finite—this assumption is consistent with the requirement that the dissipation remains finite and nonzero as the viscosity vanishes.

For convenience, we will define

$$L_0=\frac{K_0^{3/2}}{\epsilon_0}, \tag{5.33}$$

so that the dissipation ϵ at the center of the mixing layer is

$$\epsilon(0,t)=\int_0^\infty T(0,k,t)dk=\frac{K^{3/2}(t)}{L_0}, \tag{5.34}$$

where

$$\frac{1}{L_0}=\int_0^\infty h(0,\xi)d\xi. \tag{5.35}$$

The dissipation across the entire domain is now

$$\varepsilon(t)=\int_{-\infty}^{+\infty}\int_0^{\infty}\epsilon(z,t)dk\,dz$$
$$=K^{3/2}(t)\int_{-\infty}^{+\infty}\int_0^{\infty}h(\chi,\xi)d\xi\,d\chi$$
$$=K^{3/2}(t)I_h, \tag{5.36}$$

where

$$I_h=\int_{-\infty}^{+\infty}\int_0^{\infty}h(\chi,\xi)d\xi\,d\chi. \tag{5.37}$$

This is essentially the same result that one would achieve by simply stating that the total dissipation integrated across the layer has dimensions of $[L^3/T^3]$ and pursued a similar analysis.

C. Density fluctuation spectra

We present the following by first noting that we will assume the following scaling for the density:

$$\rho'=\zeta\rho \tag{5.38}$$

and restricted to

$$\zeta=\tau^{\beta}. \tag{5.39}$$

The self-similar form for the inhomogeneous spectrum of the fluctuating density-density correlation is

$$\phi_\rho(z,k,t)=B(t)W(t)f_\phi(\chi,\xi), \tag{5.40}$$

where

$$B(t)=B_0\left[\frac{t+t_0}{t_0}\right]^{2\beta} \tag{5.41}$$

and

$$\int_0^{\infty}f_\phi(0,\xi)d\xi=1. \tag{5.42}$$

The variations in density variance across the layer are

$$b(z,t)=B(t)f_b(\chi), \tag{5.43}$$

where

$$f_b(\chi)=\int_0^{\infty}f_\phi(\chi,\xi)d\xi. \tag{5.44}$$

Consider the infrared spectrum of the density-correlation spectrum, and assume a power-law spectrum wherein the exponent n is independent of z (as was done for the energy spectrum):

$$\lim_{k\to 0}\phi_\rho(z,k,t)\to\phi_0(z,t)k^n. \tag{5.45}$$

Scaling of $\phi_0(z,t)$ implies that

$$\tau^{(1+n)\gamma+2\beta}\phi_0(z,t)=\phi_0(\tau^{\gamma}z,\tau(t+t_0)-t_0). \tag{5.46}$$

This yields

$$\phi_0(z,t)=B_\phi(t)W^{1+n}(t)f_{\phi,0}(\chi), \tag{5.47}$$

where

$$B_\phi(t)=B_{\phi,0}\left[\frac{t+t_0}{t_0}\right]^{2\beta}. \tag{5.48}$$

A reasonable expectation is constancy in time of the fluctuating density correlation at the center of the mixing layer, implying that $\beta=0$, and that $\phi_0(z,t)$ behaves as

$$\phi_0(z,t)=B_{\phi,0}W^{1+n}(t). \tag{5.49}$$

The dissipation of density (or scalar) variance $\epsilon_{\rho'}$ has dimensions of ρ^2/T and thus

$$\tau^{2\beta-1}\epsilon_{\rho'}(z,t)=\epsilon_{\rho'}(\tau^{\gamma}z,\tau(t-t_0)+t_0), \tag{5.50}$$

yielding

$$(2\beta-1)\epsilon_{\rho'}=\gamma x\frac{\partial\epsilon_{\rho'}}{\partial z}+(t+t_0)\frac{\partial\epsilon_{\rho'}}{\partial t}. \tag{5.51}$$

The solution of this has the form

$$\epsilon_{\rho'}(z,t)=\epsilon_{\rho',0}\left[\frac{t+t_0}{t_0}\right]^{(2\beta-1)}f_{\epsilon\rho'}(\chi), \tag{5.52}$$

or, more conveniently,

$$\epsilon_{\rho'}(z,t)=\tilde{\epsilon}_{\rho',0}\frac{dB(t)}{dt}f_{\epsilon\rho'}(\chi). \tag{5.53}$$

D. Time-scale statistics

Consider a spectral time scale with dimensions $[LT]$:

$$(\gamma+1)\theta(z,k,t)=\frac{\partial\theta(z,k,t)}{\partial z}\frac{\gamma z}{t+t_0}-\frac{\partial\theta(z,k,t)}{\partial k}\frac{\gamma k}{t+t_0}+\frac{\partial\theta(z,k,t)}{\partial t}, \tag{5.54}$$

so that

$$\theta(z(t),k(t),t)=\theta_0\left[\frac{t+t_0}{t_0}\right]^{\gamma+1} \tag{5.55}$$

and

$$\theta(z,k,t)=\left[\frac{W^2(t)}{K^{1/2}(t)}\right]f_\theta(\chi,\xi). \tag{5.56}$$

Let

$$\theta_0=\left[\frac{W_0}{K_0^{1/2}}\right] \tag{5.57}$$

and

$$\Theta(t)=\theta_0\left[\frac{W(t)}{K^{1/2}(t)}\right]=\theta_0\left[\frac{t+t_0}{t_0}\right], \tag{5.58}$$

so

$$\theta(z,k,t)=\Theta(t)W(t)f_\theta(\chi,\xi). \tag{5.59}$$

Integrating over all wave numbers yields

$$\vartheta(z,t)=\Theta(t)\int_0^\infty f_\theta(\chi,\xi)d\xi=\Theta(t)f_\theta(\chi), \tag{5.60}$$

where we have again chosen the normalization

$$\int_0^\infty f_\theta(0,\xi)d\xi=1. \tag{5.61}$$

Note that the spectrally integrated time scale grows as a linear function of time independent of the value of γ.

E. Scaling of turbulent viscosity/diffusion

Consider a spectral diffusivity with dimensions $[L^3T^{-1}]$,

$$(3\gamma-1)\nu(z,k,t)=\frac{\partial\nu(z,k,t)}{\partial z}\frac{\gamma z}{t+t_0}-\frac{\partial\nu(z,k,t)}{\partial k}\frac{\gamma k}{t+t_0}+\frac{\partial\nu(z,k,t)}{\partial t}, \tag{5.62}$$

so

$$\nu(z(t),k(t),t)=\nu_0\left[\frac{t+t_0}{t_0}\right]^{3\gamma-1} \tag{5.63}$$

and

$$\nu(z,k,t)=K^{1/2}(t)W^2(t)Y_\nu(\chi,\xi). \tag{5.64}$$

For the particular element $\gamma=2$ we will set

$$\nu_0=K_0^{1/2}W_0, \tag{5.65}$$

so that

$$\nu(t)=\nu_0\left[\frac{t+t_0}{t_0}\right]^3 \tag{5.66}$$

and

$$\nu(z,k,t)=\nu(t)W(t)Y_\nu(\chi,\xi). \tag{5.67}$$

Integrating over all wave numbers yields

$$\nu(z,t)=\nu(t)\int_0^\infty Y_{nu}(\chi,\xi)d\xi=\nu(t)Y(\chi). \tag{5.68}$$

So the spectrally integrated turbulent viscosity grows as a cubic function of time.

VI. CONCLUSIONS

The fundamental mathematical basis for the self-similarity of KH and RT mixing layers has been demonstrated using a simplified group-theoretic approach. The fundamental assumption inherent in the assertion of full self-similarity is that the system evolves to a solution that is invariant under an appropriate subgroup of the full group of transformations under which the dynamics of the system are invariant. Pragmatically, this subgroup has the property that the "scaled" function of unscaled variables is equal to the unscaled function of scaled variables. This approach does not demonstrate that a real physical system, e.g., an experiment, or a mathematical model (e.g., Navier-Stokes or a turbulence model) will tend toward such a state. Rather, it elucidates the features of such a state, whether or not that state is physically attainable.

The linear-in-time growth rate of the KH mixing layer is shown to be a consequence of the assumption that the scaling subgroup leaves the fundamental driving mechanism of the layer, the velocity difference, invariant. Likewise, the quadratic-in-time growth rate of the RT layer is a consequence of the assumption that the acceleration is invariant. The functional dependencies of the growth-rate coefficients were obtained by exploiting the functional representations suggested by the simplified group-theoretic analysis as well as a simple energy balance. These results do not demonstrate that the growth-rate coefficients [e.g., $\alpha(At)$ for the RT layer] are "universal"—i.e., identical for different realizations of similar flow configurations. Such detailed numerical results require the use of a specific theory or model, and the universality of the coefficients may very well be model dependent. It is hoped that the relationships put forth in this paper will provide additional objective tests for full self-similarity of the Rayleigh-Taylor and Kelvin-Helmholtz mixing layers.

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